Function and Limits

- $\lim_{x\to\pm\infty}\frac{Ax^\alpha}{Bx^\beta}\begin{cases}0 & \text{if}\alpha<\beta\\ \frac{A}{B} & \text{if}\alpha=\beta\\ \pm\infty & \text{if}\alpha>\beta\end{cases}$
- $\lim_{x\to c}\frac{sin(g(x))}{g(x)}=1(\lim_{x\to c}g(x)=0)$
- $\lim_{x\to c}\frac{tan(g(x))}{g(x)}=1$
- $\lim_{x\to 0}\frac{sin(x)}{x}=1$
- $\lim_{x\to 0}\frac{tan(x)}{x}=1$
- $\lim_{n\to\infty}\frac{n!}{n^n}=0$

Differentiation

parametric differentiaton: $\frac{d^2y}{dx^2}=\frac{d}{dx}(\frac{dy}{dx})=\frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$

$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x\cot x$
$\sec x$	$\sec x\tan x$
$\cot x$	$-\csc^2 x$
$a^{f(x)}$	$\ln a\cdot f'(x)a^{f(x)}$
$\log_a f(x)$	$\log_a e\cdot\frac{f'(x)}{f(x)}$
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}},\quad \lvert f(x)\rvert<1$
$\cos^{-1} f(x)$	$-\frac{f'(x)}{\sqrt{1-[f(x)]^2}},\quad \lvert f(x)\rvert<1$
$\tan^{-1} f(x)$	$\frac{f'(x)}{1+[f(x)]^2}$
$\cot^{-1} f(x)$	$-\frac{f'(x)}{1+[f(x)]^2}$
$\sec^{-1} f(x)$	$\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$
$\csc^{-1} f(x)$	$-\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$

Second Derivative Test: $f'(c)=0, f''(c)<0$ then local max, $f''(c)>0$ local min.

L'Hopital's Rule: Given $\lim_{x\to c}f(x)$ and $g(x)=0$ or $\pm\infty$

$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$

- Use for $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Trigo Identities

1. $\sec^2 x-1=\tan^2 x$
2. $\csc^2 x-1=\cot^2 x$
3. $\sin A\cos A=\frac{1}{2}\sin 2A$
4. $\cos^2 A=\frac{1}{2}(1+\cos 2A)$
5. $\sin^2 A=\frac{1}{2}(1-\cos 2A)$
6. $\sin A\cos B=\frac{1}{2}(\sin(A+B)+\sin(A-B)$
7. $\cos A\sin B=\frac{1}{2}(\sin(A+B)-\sin(A-B)$
8. $\cos A\cos B=\frac{1}{2}(\cos(A+B)+\cos(A-B)$
9. $\sin A\sin B=\frac{1}{2}(\cos(A+B)-\cos(A-B)$

Integration

$f(x)$	$\int f(x)$
$\tan ax$	$\frac{1}{a}\ln\lvert\sec(ax)\rvert$
$\cot ax$	$\frac{1}{a}\ln\lvert\cot(ax)\rvert$
$\sec ax$	$\frac{1}{a}\ln\lvert\sec(ax)+tan(ax)\rvert$
$\csc ax$	$\frac{1}{a}\ln\lvert\csc(ax)+cot(ax)\rvert$
$\frac{1}{a^2+(x+b)^2}$	$\frac{1}{a}\tan^{-1}(\frac{x+b}{a})$
$\frac{1}{\sqrt{a^2-(x+b)^2}}$	$\sin^{-1}(\frac{x+b}{a})$
$\frac{1}{a^2-(x+b)^2}$	$\frac{1}{2a}\ln\lvert\frac{x+b+a}{x+b-a}\rvert$
$\frac{1}{(x+b)^2-a^2}$	$\frac{1}{2a}\ln\lvert\frac{x+b-a}{x+b+a}\rvert$

Substitution $\int f(g(x))\cdot g'(x)dx=\int f(u)du, u=g(x)$
By Parts $\int uv'dx=uv-\int u'vdx$, order: LIATE: Differentiate to integrate

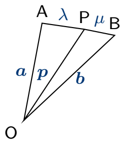
Application of Integration

about x axis

- Vol Disk: $V=\pi\int_a^b f(x)^2-g(x)^2dx$
- Vol Shell: $V=2\pi\int_a^b x\lvert f(x)-g(x)\rvert dx$ (absolute!!)
- Length of curve: $\int_a^b\sqrt{1+f'(x)^2}dx$

Vectors

unit vector: $\hat{p}=\frac{p}{\lvert p\rvert}$, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$



ratio theorem
$\boldsymbol{p}=\frac{\mu\boldsymbol{a}+\lambda\boldsymbol{b}}{\lambda+\mu}$

midpoint theorem
$\boldsymbol{p}=\frac{\boldsymbol{a}+\boldsymbol{b}}{2}$

Dot Product

- $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3=\lvert a\rvert\lvert b\rvert\cos\theta$
- $a\perp b\Rightarrow a\cdot b=0$
- $a\parallel b\Rightarrow a\cdot b=\lvert a\rvert\lvert b\rvert$

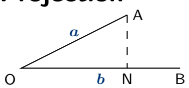
Cross Product

$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\a_i&a_2&a_3\\b_i&b_2&b_3\end{vmatrix}=\begin{pmatrix}(a_2b_3-a_3b_2)\\-(a_1b_3-a_3b_1)\\(a_1b_2-a_2b_1)\end{pmatrix}$

$\lvert\boldsymbol{a}\times\boldsymbol{b}\rvert=\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\sin\theta$
$a\perp b\Rightarrow a\times b=\lvert a\rvert\lvert b\rvert$
$a\parallel b\Rightarrow a\times b=0$

Area Parallelogram $=\lvert\boldsymbol{a}\times\boldsymbol{b}\rvert$

Projection



$\triangle ANO=\frac{1}{2}\lvert\overrightarrow{OA}\times\overrightarrow{ON}\rvert$

comp$_{\boldsymbol{b}}\boldsymbol{a}=\lvert\boldsymbol{b}\rvert\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\lvert\boldsymbol{a}\rvert}$
proj$_{\boldsymbol{b}}\boldsymbol{a}=$ comp$_{\boldsymbol{b}}\boldsymbol{a}\cdot\frac{a}{\lvert a\rvert}=$
$\overrightarrow{ON}=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\lvert\boldsymbol{a}\rvert^2}\boldsymbol{b}$

Lines

$\boldsymbol{r}=\boldsymbol{r}_0+t\boldsymbol{v}=\langle x,y,z\rangle$
$\langle x_0,y_0,z_0\rangle+t\langle a,b,c\rangle$

$\begin{pmatrix}x_0+at\\y_0+bt\\z_0+ct\end{pmatrix}$

Planes

$\boldsymbol{n}=\langle a,b,c\rangle,\boldsymbol{r}=\langle x,y,z\rangle,\boldsymbol{r}_0\langle x_0,y_0,c_0\rangle$
Scalar: $\boldsymbol{n}\cdot\boldsymbol{r}=\boldsymbol{n}\cdot\boldsymbol{r}_0$
Cartesian: $ax+by+cz=d$

Distance from Point to Plane

$\frac{\lvert ax_0+by_0+cz_0-d\rvert}{\sqrt{a^2+b^2+c^2}}$

Partial Derivatives

Chain Rule

For $z(t)=f(x(t),y(t))$,
$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$

For $z(s,t)=f(x(s,t),y(s,t))$,
$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$
$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

Arc Length of $r(t)$: $\int_a^b\lvert\boldsymbol{r}'(t)\rvert dt$

Implicit Differentiation

$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}\ \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$

Directional Derivative

Gradient vector at $f(x,y):\nabla f=f_x\boldsymbol{i}+f_y\boldsymbol{j}$
$D_uf(x,y)=\langle f_x,f_y\rangle\cdot\langle a,b\rangle=\langle f_x,f_y\rangle\cdot\hat{\boldsymbol{u}}=\ \nabla f\cdot\hat{\boldsymbol{u}}$
(Unit Vector)
Tangent Plane: $\langle f_x,f_y-1\rangle\cdot\langle x-x_0,y-y_0,z-z_0\rangle=0$

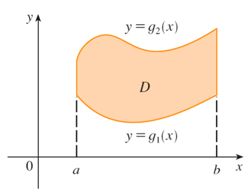
Critical Points

$f_x=0$ and $f_y=0$, OR (f_x or f_y does not exist)
$D=f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2$

D	$f_{xx}(a,b)$	**local**
+	+	min
+	-	max
-	any	saddle point
0	any	no conclusion

Double Integrals

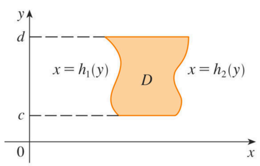
Type I



$\int_a^b\int_{g_1(x)}^{g_2(x)}f(x,y)dydx$

$D=\{(x,y):a\le x\le b, g_1(x)\le y\le g_2(x)\}$

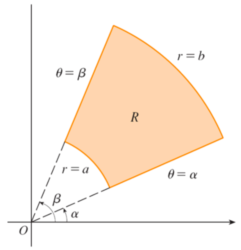
Type II



$\int_c^d\int_{h_1(y)}^{h_2(y)}f(x,y)dxdy$

$D=\{(x,y):c\le y\le d, h_1(y)\le x\le h_2(y)\}$

Polar Coordinates



$x=r\cos\theta$
$y=r\sin\theta$
$R=\{(r,\theta):0\le a\le r\le b,\ \alpha\le\theta\le\beta\}$

$\int_\alpha^\beta\int_a^b f(r\cos\theta,r\sin\theta)rdrd\theta$

Surface Area

$S=\iint_R\sqrt{f_x^2+f_y^2+1}dA$, get in the form of $z=f(x,y)$ first

ODE

form	change of variable
$\frac{dy}{dx}=f(x)g(y)$	$\int\frac{1}{g(y)}dy=\int f(x)dx+C$
$y'=g(\frac{y}{x})$	Set $v=\frac{y}{x}$ $\Rightarrow y'=v+xv'$
$y'=f(ax+by+c)$ $\Rightarrow y'=\frac{ax+by+c}{\alpha x+\beta y+\gamma}$	Set $v=ax+by$
$y'+P(x)y=Q(x)$	$R=e^{\int P(x)dx}$ $\Rightarrow y\cdot R=\int Q\cdot Rdx$
$y'+P(x)y=Q(x)y^n$	$z=y^{1-n}$ $\Rightarrow$ sub in Z solve linear

Population Models

$N_\infty=\frac{B}{s}$, $\hat{N}=$ Population Now

Malthus
$N(t)=\hat{N}e^{kt}$
$k=B-D$

Logistic
$\frac{1}{N}=\frac{1}{N_\infty}+(\frac{1}{\hat{N}}-\frac{1}{N_\infty})e^{-Bt}$
$N=\frac{N_\infty}{1+(\frac{N_\infty}{\hat{N}}-1)e^{-Bt}}$

Uranium Decay into Thorium

$U(t)=U_0e^{-k_ut}$, $k=\frac{\ln 2}{\text{halflife}}$, $\frac{dU}{dt}=-k_uU$
Thorium:
$T(t)=\frac{K_uU_0}{K_t-K_u}(e^{-k_ut}-e^{-k_tt})$, $\frac{dT}{dt}=k_uU-k_TT$

Series

Geometric Series

$\sum_{n=1}^{\infty} ar^{n-1}, a \neq 0$ converges to $\frac{a}{1-r}$ when $|r| < 1$, diverges otherwise

If series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$

Tests

Decreasing function -> differentiate and see the range where $x < 0$

Test	Method
n^{th} term	$\lim\limits_{n\to\infty} a_n \neq 0$ or does not exist, then divergent
Integral	$f(n) = a_n$ is continuous, positive, decreasing function $\forall x \geq 1$ and $\int_1^{\infty} f(x)dx$ converges else divergent
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent $\leftrightarrow p > 1$
Harmonic Series	$\sum_{n=1}^{\infty} \frac{1}{n}$ divergent
Ratio If Factorial	$0 \geq \lim\limits_{n\to\infty} \lvert\frac{a_{n+1}}{a_n}\rvert = L < 1$ abs. convergent, > 1 divergent, $= 1$ inconclusive
Root If nth power	$0 \geq \lim\limits_{n\to\infty} \sqrt[n]{a_n} = L < 1$ abs. convergent, > 1 divergent, $= 1$ inconclusive
Alternating series	b_n decreasing, $\lim\limits_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 ...$ is convergent
Power Series	b_n decreasing, $\lim\limits_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 ...$ is convergent
Comparison Test	$\sum a_n$ and $\sum b_n$ s.t. $a_n \leq b_n$ Then if $\sum b_n$ convergent, $\sum a_n$ convergent. If $\sum a_n$ divergent, $\sum b_n$ divergent

Power Series

$\sum_{n=0}^{\infty} c_n(x-a)^n$ converges at **ONE OF**

- $x = a$
- For all x
- converges if $|x-a| < R$ and diverges if $|x-a| > R$ (R is radius of convergence)

If $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n} = L\right|$ or $\lim_{n\to\infty} \sqrt[n]{|c_n|} = L$, $L \in \mathbb{R}$ or ∞, then $R = \frac{1}{L}$

If power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then function f is differentiable on interval $|x-a| < R$ and

- $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$, for $|x-a| < R$
- $\int f(x) = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$ for $|x-a| < R$

Taylor and Maclaurin Series

If f has power series repr @ $f = a$, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, |x-a| < R, R > 0$, then $c_n = \frac{f^{(n)}(a)}{n!}$.

Maclaurin Series: $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$ For $-\infty < x < \infty$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

For $-1 < x < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \ldots$$

Useful Math

- Line: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
- $\int \sqrt{a^2 - x^2}dx = \frac{a^2}{2}\sin^{-1}(\frac{x}{a}) + \frac{x}{2}\sqrt{a^2 - x^2}, x = a\sin\theta, dx = a\cos\theta d\theta$,A
- $\sqrt{a^2 + x^2}dx = \frac{1}{2}\left(x\sqrt{a^2+x^2} + a^2 \ln\left|\frac{x+\sqrt{a^2+x^2}}{a}\right|\right), x = a\tan\theta, \frac{-\pi}{2} < \theta < \frac{\pi}{2}$
- $\int \cos^2 x = \frac{1}{4}\sin 2x + \frac{x}{2} = \frac{1}{2}\cos x \sin x + \frac{1}{2}x$
- $\int \sin^2 x = -\frac{1}{4}\sin 2x + \frac{x}{2}$
- $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$