MA1522

Linear Algebra in Computing

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Contents

1 Linear Systems

1.1 Linear Algebra

- **Linear** The study of items/planes and objects which are flat
- **Algebra** Objects are not as simple as numbers

1.2 Linear Systems & Their Solutions

Points on a straight line are all the points (x, y) on the xy plane satisfying the linear eqn: $ax + by = c$, where $a, b > 0$

1.2.1 Linear Equation

Linear eqn in n variables (unknowns) is an eqn in the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

where $a_1, a_2, ..., a_n, b$ are constants.

Note. In a linear system, we don't assume that $a_1, a_2, ..., a_n$ are not all 0

- If $a_1 = ... = a_n = 0$ but $b \neq 0$, it is **inconsistent**

  E.g. $0x_1 + 0x_2 = 1$

- If $a_1 = ... = a_n = b = 0$, it is a **zero equation**

  E.g. $0x_1 + 0x_2 = 0$

- Linear equation which is not a zero equation is a **nonzero equation**

  E.g. $2x_1 - 3x_2 = 4$

- The following are not linear equations
  - $xy = 2$
  - $\sin\theta + \cos\phi = 0.2$
  - $x_1^2 + x_2^2 + ... + x_n^2 = 1$
  - $x = e^y$

In the xyz space, linear equation $ax + by + cz = d$ where $a, b, c > 0$ represents a plane

1.2.2 Solutions to a Linear Equation

Let $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ be a linear eqn in n variables
For real numbers $s_1+s_2+...+s_n$, if $a_1s_1+a_2s_2+...+a_ns_n = b$, then $x_1 = s_1, x_2 = s_2, x_n = s_n$ is a solution to the linear equation

The set of all solutions is the **solution set**
Expression that gives the entire solution set is the **general solution**

Zero Equation is satified by any values of $x_1, x_2, ...x_n$
General solution is given by $(x_1, x_2, ..., x_n) = (t_1, t_2, ..., t_n)$

1.2.3 Examples: Linear equation $4x - 2y = 1$

- x can take any arbitary value, say t
- $x = t \Rightarrow y = 2t - \frac{1}{2}$
- General Solution: $\begin{cases} x = t & \text{t is a parameter} \\ y = 2t - \frac{1}{2} \end{cases}$
- y can take any arbitary value, say s
- $y = s \Rightarrow x = \frac{1}{2}s + \frac{1}{4}$
- General Solution: $\begin{cases} y = s & \text{s is a parameter} \\ x = \frac{1}{2}s + \frac{1}{4} \end{cases}$

1.2.4 Example: Linear equation $x_1 - 4x_2 + 7x_3 = 5$

- x_2 and x_3 can be chosen arbitarily, s and t
- $x_1 = 5 + 4s - 7t$
- General Solution: $\begin{cases} x_1 = 5 + 4s - 7t \\ x_2 = s & s, t \text{ are arbitrary parameters} \\ x_3 = t \end{cases}$

1.3 Linear System

Linear System of m linear equations in n variables is

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases} \tag{1}$$

where a_{ij}, b are real constants and a_{ij} is the coeff of x_j in the ith equation

Note. Linear Systems

- If a_{ij} and b_i are zero, linear system is called a **zero system**
- If a_{ij} and b_i is nonzero, linear system is called a **nonzero system**
- If $x_1 = s_1, x_2 = s_2, ..., x_n = s_n$ is a solution to **every equation** in the system, then its a solution to the system
- If every equation has a solution, there might not be a solution to the system
- **Consistent** if it has at least 1 solution
- **Inconsistent** if it has no solutions

1.3.1 Example

$$\begin{cases} a_1x + b1_y = c_1 \\ a_2x + b2_y = c_2 \end{cases} \tag{2}$$

where a_1, b_1, a_2, b_2 not all zero
In xy plane, each equation represents a straight line, L_1, L_2

- If L_1, L_2 are parallel, there is no solution
- If L_1, L_2 are not parallel, there is 1 solution
- If L_1, L_2 coinside(same line), there are infinitely many solution

$$\begin{cases} a_1x + b1_y + c_1z = d_1 \\ a_2x + b2_y + c_2z = d_2 \end{cases} \tag{3}$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ not all zero
In xyz space, each equation represents a plane, P_1, P_2

- If P_1, P_2 are parallel, there is no solution
- If P_1, P_2 are not parallel, there is ∞ solutions (on the straight line intersection)
- If P_1, P_2 coinside(same plane), there are infinitely many solutions
- Same Plane $\Leftrightarrow a_1 : a_2 = b_1 : b_2 = c_1 : c_2 = d_1 : d_2$
- Parallel Plane $\Leftrightarrow a_1 : a_2 = b_1 : b_2 = c_1 : c_2$
- Intersect Plane $\Leftrightarrow a_1 : a_2, b_1 : b_2, c_1 : c_2$ are not the same

1.4 Augmented Matrix

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{1n} & b_1 \\ a_{21} & a_{12} & a_{2n} & b_2 \\ a_{m1} & a_{m2} & a_{mn} & b_m \end{array}\right)$$

1.5 Elementary Row Operations

To solve a linear system we perform operations:

- Multiply equation by nonzero constant
- Interchange 2 equations
- add a constant multiple of an equation to another

Likewise, for a augmented matrix, the operations are on the **rows** of the augmented matrix

- Multiply row by nonzero constant
- Interchange 2 rows
- add a constant multiple of a row to another row

1.6 Recap

Given the linear equation $a_1x_1 + a_2x_2 + ... + a_nx_n = b$

1. $a_1 = a_2 = ... = a_n = b = 0$ zero equation

   Solution: $x_1 = t_1, x_2 = t_2, ... = x_n = t_n$

2. $a_1 = a_2 = ... = a_n = 0 \neq b$ inconsistent

   No Solution

3. Not all $a_1...a_n$ are zero.

   Set $n - 1$ of x_i as params, solve for last variable

1.7 Elementary Row Operations Example

$$\begin{cases} x + \ \ y + 3z = 0 \\ 2x - 2y + 2z = 4 \\ 3x + 9y \qquad = 3 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

1.8 Row Equivalent Matrices

2 Augmented Matrices are row equivalent if one can be obtained from the other by a series of elementary row operations

Given a augmented matrix A, how to find a row equivalent augmented matrix B of which is of a **simple** form?

1.9 Row Echelon Form

Definition (Row Echelon Form (Simple))**.** Augmented Matrix is in row-echelon form if

- Zero rows are grouped together at the bottom
- For any 2 successive nonzero rows, The first nonzero number in the lower row appears to the right of the first nonzero number on the higher row $\left(\begin{array}{cccc|c} 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$
- Leading entry if a nonzero row is a **pivot point**
- Column of augmented matrix is called
  - **Pivot Column** if it contains a pivot point
  - **Non Pivot Column** if it contains no pivot point
- Pivot Column contains exactly 1 pivot point

  # of pivots = # of leading entries = # of nonzero rows

Examples of row echlon form:

$$\left(\begin{array}{cc|c} 3 & 2 & 1 \end{array}\right) \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right) \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{array}\right) \left(\begin{array}{cccc|c} 0 & 1 & 2 & 8 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Examples of NON row echlon form:

$$\left(\begin{array}{cc|c} 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 \end{array}\right) \left(\begin{array}{cc|c} 0 & 0 & \mathbf{1} \\ \mathbf{1} & -1 & 0 \\ 0 & 0 & 1 \end{array}\right) \left(\begin{array}{ccc|c} \mathbf{1} & 0 & 2 & 1 \\ 0 & \mathbf{1} & 0 & 2 \\ 0 & \mathbf{1} & 1 & 3 \end{array}\right) \left(\begin{array}{cccc|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

1.10 Reduced Row-Echelon Form

Definition (Reduced Row-Echelon Form)**.** Suppose an augmented matrix is in row-echelon form. It is in **reduced row-echelon form** if

- Leading entry of every nonzero row is 1

  Every pivot point is one
- In each pivot column, except the pivot point, all other entries are 0.

Examples of reduced row-echelon form:

$$\left(\begin{array}{cc|c}1 & 2 & 3\end{array}\right) \left(\begin{array}{cc|c}0 & 0 & 0\\0 & 0 & 0\end{array}\right) \left(\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right) \left(\begin{array}{ccc|c}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & 3\end{array}\right) \left(\begin{array}{cccc|c}0 & 1 & 2 & 0 & 1\\0 & 0 & 0 & 1 & 3\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right)$$

Examples of row-echelon form but not reduced: (pivot point is not 1 / all other elements **in pivot column** must be zero)

$$\left(\begin{array}{cc|c}\mathbf{3} & 2 & 1\end{array}\right) \left(\begin{array}{cc|c}1 & \mathbf{-1} & 0\\0 & 1 & 0\end{array}\right) \left(\begin{array}{cc|c}\mathbf{2} & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right) \left(\begin{array}{ccc|c}\mathbf{-1} & 2 & 3 & 4\\0 & 1 & 1 & 2\\0 & 0 & 2 & 3\end{array}\right) \left(\begin{array}{cccc|c}0 & 1 & 2 & \mathbf{8} & 1\\0 & 0 & 0 & \mathbf{4} & 3\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right)$$

To note: 2nd matrix has -1 in the pivot column, but 5th matrix has 2 in a non-pivot column so its fine

1.11 Solving Linear System

If Augmented Matrix is in reduced row-echelon form, then solving it is easy

$$\left(\begin{array}{ccc|c}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & 3\end{array}\right) \text{then } x_1 = 1, x_2 = 2, x_3 = 3$$

> **Note.** • If any equations in the system is inconsistent, the whole system is inconsistent

1.11.1 Examples

Augmented Matrix: $\left(\begin{array}{cccc|c}1 & -1 & 0 & 3 & -2\\0 & 0 & 1 & 2 & 5\\0 & 0 & 0 & 0 & 0\end{array}\right)$

- The zero row can be ignored. $\begin{cases} x_1 - x_2 \quad + 3x_4 = -2 \\ \qquad x_3 + 2x_4 = 5 \end{cases}$
- Degree of freedom(# cols): 4, number of restrictions (# pivot cols): 2, arbitrary vars(# non pivot cols): 4-2 = 2. Set this to the non-pivot cols

1. Let $x_4 = t$ and sub into 2nd eqn

   $x_3 + 2t = 5 \Rightarrow x_3 = 5 - 2t$

2. sub $x_4 = t$ into 1st eqn

   $x_1 - x_2 + 3t = -2$

   Let $x_2 = s$. Then $x_1 = -2 + s - 3t$

3. Infinitely many sols with (s and t as arbitrary params)

   $x_1 = -2 + s - 3t, x_2 = s, x_3 = 5 - 2t, x_4 = t$

Augmented Matrix: $\left(\begin{array}{ccccc|c} 0 & 2 & 2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{array}\right)$

- $\begin{cases} 0x_1 + 2x_2 + 2x_3 + 1x_4 - 2x_5 = 2 \\ x_3 + x_4 + x_5 = 3 \\ 2x_5 = 4 \end{cases}$
- Degree of freedom: 5, number of restrictions: 3, arbitrary vars: 5-3 = 2

1. by 3rd eqn, $2x_5 = 4 \Rightarrow x_5 = 2$
2. sub $x_5 = 2$ into 2nd eqn

   $x_3 + x_4 + 2 = 3 \Rightarrow x_3 + x_4 = 1$

   let $x_4 = t$. Then $x_3 = 1 - t$
3. sub $x_5 = 2, x_3 = 1 - t, x_4 = t$ into 1st eqn

   $2x_2 + 2(1 - t) + t - 2(2) = 2 \Rightarrow 2x_2 - t = 4 \Rightarrow x_2 = \frac{t}{2} + 2$
4. system has inf many solns: $x_1 = s, x_2 = \frac{t}{2} + 2, x_3 = 1 - t, x_4 = t, x_5 = 2$ where s and t are arbitrary

1.11.2 Algorithm

Given the augmented matrix is in row-echelon form.

1. Set variables corresponding to non-pivot columns to be arbitrary parameters
2. Solve variables corresponding to pivot columns by back substitution (from last eqn to first)

1.12 Gaussian Eliminiation

Definition (Gaussian Elimination)**.**

1. Find the left most column which is not entirely zero
2. If top entry of such column is 0, replace with nonzero number by swapping rows
3. For each row below top row, add multiple of top row so that leading entry becomes 0
4. Cover top row and repeat to remaining matrix

Note (Algorithm with Example).

$$\left(\begin{array}{cccccc|c} 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right)$$

1. Find the left most column which is not all zero (2nd column)

2. Check top entry of the selection. If its zero, replace it by a nonzero number by interchanging the top row with another row below

$$\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right)$$

3. For each row below the top row, adda suitable multiple of top row so that leading entry becomes 0.

   $2R_1 + R_3$ will ensure that the -2 turns to 0

$$\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 3 & 6 & 9 & -12 \end{array}\right)$$

4. Cover top row and repeat procedure to the remaining matrix

$$\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ \hline 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 3 & 6 & 9 & -12 \end{array}\right)$$

   Look at C_4. $R_3 \times -1.5R_2$ will set R_3C_4 to zero.

$$\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ \hline 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right)$$

   This is now in row echelon form.

Only use $R_i \Leftrightarrow R_j or R_i + CR_j$ in this method.

1.13 Gauss-Jordan Elimination

Definition (Gauss Joran Elimination).

1-4. Use Gaussian Eliminiation to get row-echelon form

5. For each nonzero row, multiply a suitable constant so pivot point becomes 1

6. Begin with last nonzero row and work backwords

   Add suitable multiple of each row to the rows above to introduce 0 above pivot point

- Every matrix has a unique reduced row-echelon form.
- Every nonzero matrix has infinitely many row-echelon ofrm

Note (Gauss Jordan Elimination Example). Suppose an augmented matrix is in row-echelon form. $\left(\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right)$

1. All pivot points must be 1

   multiply R_2 by $\frac{1}{2}$ and R_3 by $\frac{1}{6}$

   $\left(\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$

2. In each pivot col, all entries other than pivot point must be 0. Work backwards

   $R_1 + -3R_1$, $R_2 + -R_1$

   $\left(\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$

   $R_1 + -4R_2$

   $\left(\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 0 & -29 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$

1.14 Review

$$I : cR_i, c \neq 0$$
$$II : R_i \Leftrightarrow R_j$$
$$III : R_i \Rightarrow R_i + cR_j$$

Solving REF:

1. Set var -> non-pivot cols as params
2. Solve var -> pivot cols backwards

   $\#$ of nonzero rows $= \#$ pivot pts $= \#$ of pivot cols

Gaussian Elimination

1. Given a matrix A, find left most non-zero **column**. If the leading number is NOT zero, use II to swap rows.
2. Ensure the rest of the column is 0 (by subtracting the current row from tht other rows)
3. Cover the top row and continue for next rows

1.15 Consistency

Definition (Consistency)**.**
Suppose that A is the Augmented Matrix of a linear system, and R is a row-echelon form of A.

- When the system has no solution(inconsistent)?

  There is a row in R with the form $(00...0|\otimes)$ where $\otimes \neq 0$

  Or, the last column is a pivot column

- When the system has exactly one solution?

  Last column is non-pivot

  All other columns are pivot columns

- When the system has infinitely many solutions?

  Last column is non-pivot

  Some other columns are non-pivot columns.

Note. Notations
For elementary row operations

- Multiply ith row by (nonzero) const k: kR_i
- Interchange ith and jth rows: $R_i \leftrightarrow R_j$
- Add K times ith row to jth row: $R_j + kR_i$

Note

- $R_1 + R_2$ means "add 2nd row to the 1st row".
- $R_2 + R_1$ means "add 1nd row to the 2st row".

Example

$$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} a+b \\ b \end{pmatrix} \xrightarrow{R_2+(-1)R_1} \begin{pmatrix} a+b \\ -a \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} b \\ -a \end{pmatrix} \xrightarrow{(-1)R_2} \begin{pmatrix} b \\ a \end{pmatrix}$$

1.16 Homogeneous Linear System

Definition (Homogeneous Linear Equation & System)**.** where

- Homogeneous Linear Equation: $a_1x_1 + a_2x_2 + ... + a_nx_n = 0 \Leftrightarrow x_1 = 0, x_2 = 0, ..., x_n = 0$
- Homogeneous Linear Equation: $\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0 \end{cases}$
- This is the trivial solution of a homogeneous linear system.

You can use this to solve problems like Find the equation $ax^2 + by^2 + cz^2 = d$, in the xyz plane which contains the points $(1, 1, -1), (1, 3, 3), (-2, 0, 2)$.

- Solve by first converting to Augmented Matrix, where the last column is all 0. During working steps, this column can be omitted.
- With the RREF, you can set d as t and get values for a, b, c in terms of t.
- sub in t into the original equation and factorize t out from both sides, for values where $t \neq 0$

2 Matrices

2.1 Introduction

Definition (Matrix).

- $$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$
- m is no of rows, n is no of columns
- size is $m \times n$
- $A = (a_{ij})_{m \times n}$

2.1.1 Special Matrix

Note (Special Matrices).

- Row Matrix : $\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$
- Column Matrix
  $$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$
- **Square Matrix**, $n \times n$ matrix / matrix of order n.

  Let $A = (a_{ij})$ be a square matrix of order n
- Diagonal of A is $a_{11}, a_{22}, ..., a_{nn}$.
- **Diagonal Matrix** if Square Matrix and non-diagonal entries are zero

  Diagonals can be zero

  Identity Matrix is a special case of this
- **Square Matrix** if Diagonal Matrix and diagonal entries are all the same.
- **Identity Matrix** if Scalar Matrix and diagonal $= 1$

  I_n is the identity matrix of order n.
- **Zero Matrix** if all entries are 0.

  Can denote by either $\overrightarrow{0}, 0$
- Square matrix is **symmetric** if symmetric wrt diagonal

  $A = (a_{ij})_{n \times n}$ is symmetric $\Leftrightarrow a_{ij} = a_{ji},\ \forall i, j$
- **Upper Triangular** if all entries **below** diagonal are zero.

  $A = (a_{ij})_{n \times n}$ is upper triangular $\Leftrightarrow a_{ij} = 0$ if $i > j$
- **Lower Triangular** if all entries **above** diagonal are zero.

  $A = (a_{ij})_{n \times n}$ is lower triangular $\Leftrightarrow a_{ij} = 0$ if $i < j$

  if Matrix is both Lower and Upper triangular, its a Diagonal Matrix.

2.2 Matrix Operations

Definition (Matrix Operations).
Let $A = (a_{ij})_{m\times n}, B = (b_{ij})_{m\times n}$

- Equality: $B = (b_{ij})_{p\times q}$, $A = B \Leftrightarrow m = p \;\&\; n = q \;\&\; a_{ij} = b_{ij} \forall i, j$
- Addition: $A + B = (a_{ij} + b_{ij})_{m\times n}$
- Subtraction: $A - B = (a_{ij} - b_{ij})_{m\times n}$
- Scalar Mult: $cA = (ca_{ij})_{m\times n}$

Definition (Matrix Multiplication).
Let $A = (a_{ij})_{m\times p}, B = (b_{ij})_{p\times n}$

- AB is the $m \times n$ matrix s.t. (i, j) entry is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

- No of columns in A = No of rows in B.
- Matrix multiplication is **NOT commutative**

Theorem 2.1 (Matrix Properties).
Let A, B, C be $m \times p, p \times q, q \times n$ matrices

- Associative Law: $A(BC) = (AB)C$
- Distributive Law: $A(B_1 + B_2) = AB_1 + AB_2$
- Distributive Law: $(B_1 + B_2)A = B_1A + B_2A$
- $c(AB) = (cA)B = A(cB)$
- $A\mathbf{0}_{p\times n} = \mathbf{0}_{m\times n}$
- $A\mathbf{I}_n = \mathbf{I}_nA = A$

Definition (Powers of Square Matricss)**.**
Let A be a $m \times n$.
AA is well defined $\Leftrightarrow m = n \Leftrightarrow A$ is square.
Definition. Let A be square matrix of order n. Then Powers of a are

$$A^k = \begin{cases} I_n & \text{if } k = 0 \\ AA...A & \text{if } k \geq 1. \end{cases}$$

Properties.

- $A^m A^n = A^{m+n}, (A^m)^n = A^{mn}$
- $(AB)^2 = (AB)(AB) \neq A^2 B^2 = (AA)(BB)$

Matrix Multiplication Example:

- Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}$
- Let $a_1 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, a_2 = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$
- $AB = \begin{pmatrix} a_1 & a_2 \end{pmatrix} B = \begin{pmatrix} a_1 B \\ a_2 B \end{pmatrix}$.
- $\begin{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} \\ \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix} \\ \begin{pmatrix} 8 & 7 \end{pmatrix} \end{pmatrix}$

Note (Representation of Linear System)**.**

- $\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n & = b_2 \\ \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n & = b_m \end{cases}$
- A $= \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$, Coefficient Matrix, $A_{m \times n}$
- $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, Variable Matrix, $x_{n \times 1}$
- $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$, Constant Matrix, $b_{m \times 1}$. Then $Ax = b$
- $A = (a_{ij})_{m \times n}$
- m linear equations in n variables, $x_1, ..., x_n$
- a_{ij} are coefficients, b_i are the constants
- Let $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$.

  $x_1 = u_1, \ldots, x_n = u_n$ is a solution to the system

  $\Leftrightarrow Au = b \Leftrightarrow u$ is a solution to $Ax = b$
- Let a_j denote the jth column of A. Then

  $b = Ax = x_1 a_1 + ... + x_n a_n = \sum_{j=1}^{n} x_j a_j$

Definition (Transpose).

- Let $A = (a_{ij})_{m\times n}$
- The transpose of A is $A^T = (a_{ji})_{n\times m}$
- $(A^T)^T = A$
- A is symmetric $\Leftrightarrow A = A^T$
- Let B be $m \times n$, $(A+B)^T = A^T + B^T$
- Let B be $n \times p$, $(AB)^T = B^T A^T$

Definition (Inverse).

- Let A, B be matrices of same size

  $A + X = B \Rightarrow X = B - A = B + (-A)$

  $-A$ is the *additive inverse* of A

- Let $A_{m\times n}, B_{m\times p}$ matrix.

  $AX = B \Rightarrow X = A^{-1}B.$

Let A be a **square matrix** of order n.

- If there exists a square matrix B of order N s.t. $AB = I_n$ and $BA = I_n$, then A is **invertible** matrix and B is inverse of A.
- If A is not invertible, A is called singular.
- suppose A is invertible with inverse B
- Let C be any matrix having the same number of rows as A.

$$\begin{aligned} AX = C \Rightarrow B(AX) &= BC \\ \Rightarrow (BA)X &= BC \\ \Rightarrow X &= BC. \end{aligned}$$

Theorem 2.2 (Properties of Inversion)**.**
Let A be a square matrix.

- Let A be an invertible matrix, then its inverse is unique.
- Cancellation Law: Let A be an invertible matrix

  $AB_1 = AB_2 \Rightarrow B_1 = B_2$

  $C_1A = C_2A \Rightarrow C_1 = C_2$

  $AB = 0 \Rightarrow B = 0, CA = 0 \Rightarrow C = 0$ (A is invertible, A cannot be 0)

  This fails if A is singular
- Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

  A is invertible $\Leftrightarrow ad - bc \neq 0$

  A is invertible $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Let A and B be invertible matrices of same order

- Let $c \neq 0$. Then cA is invertible, $(cA^{-1} = \frac{1}{c}A^{-1}$
- A^T is invertible, $(A^T)^{-1} = (A^{-1})^T$
- AB is invertible, $(AB)^{-1} = (B^{-1}A^{-1})$

Let A be an invertible matrix.

- $A^{-k} = (A^{-1})^k$
- $A^{m+n} = A^mA^n$
- $(A^m)^n = A^{mn}$

Definition (Elementary Matrices)**.** If it can be obtained from I by performing single elementary row operation

- $cRi, c \neq 0:$ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (cR_3)$
- $R_i \leftrightarrow R_j, i \neq j, :$ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} (R_2 \leftrightarrow R_4)$
- $R_i + cR_j, i \neq j, :$ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (R_2 + cR_4)$
- Every elementary Matrix is invertible

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}, \ E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (cR_3), \ EA = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ca_{31} & ca_{32} & ca_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

Theorem 2.3. Main Theorem for Invertible Matrices
Let A be a square matrix. Then the following are equivalent

1. A is an invertible matrix.
2. Linear System $Ax = b$ has a unique solution
3. Linear System $Ax = 0$ has only the trivial solution
4. RREF of A is I
5. A is the product of elementary matrices

Theorem 2.4. Find Inverse

- Let A be an invertible Matrix.
- RREF of $(A|I)$ is $(I|A^{-1})$

How to identify if Square Matrix is invertible?

- Square matrix is invertible

  $\Leftrightarrow$ RREF is I

  $\Leftrightarrow$ All columns in its REF are pivot

  $\Leftrightarrow$ All rows in REF are nonzero

- Square matrix is singular

  $\Leftrightarrow$ RREF is **NOT** I

  $\Leftrightarrow$ Some columns in its REF are non-pivot

  $\Leftrightarrow$ Some rows in REF are zero.

- A and B are square matrices such that $AB = I$

  then A and B are invertible

Definition (LU Decomposition with Type 3 Operations)**.**

- Type 3 Operations: $(R_i + cR_j, i > j)$
- Let A be a $m \times n$ matrix. Consider Gaussian Elimination $A \dashrightarrow R$
- Let $R \dashrightarrow A$ be the operations in reverse
- Apply the same operations to $I_m \dashrightarrow L$. Then $A = LR$
- L is a lower triangular matrix with 1 along diagonal
- If A is square matrix, $R = U$

Application:

- A has LU decomposition $A = LU$, $Ax = b$ i.e., $LUx = b$
- Let $y = Ux$, then it is reduced to $Ly = b$
- $Ly = b$ can be solved with forward substitution.
- $Ux = y$ is the REF of A.
- $Ux = y$ can be solved using backward substitution.

Definition (LU Decomposition with Type II Operations).

- Type 2 Operations: $(R_i \leftrightarrow R_j)$, where 2 rows are swapped
- $A \xrightarrow{E_1} \bullet \xrightarrow{E_2} \bullet \xrightarrow[E_3]{R_i \Leftrightarrow R_j} \bullet \xrightarrow{E_4} \bullet \xrightarrow{E_5} R$
- $A = E_1^{-1}E_2^{-1}E_3E_4^{-1}E_5^{-1}R$
- $E_3A = (E_3E_1^{-1}E_2^{-1}E_3)E_4^{-1}E_5^{-1}R$
- $P = E_3, L = (E_3E_1^{-1}E_2^{-1}E_3)E_4^{-1}E_5^{-1}, R = U,\ PA = LU$

Definition (Column Operations).

- Pre-multiplication of Elementary matrix $\Leftrightarrow$ Elementary row operation

  $A \to B \Leftrightarrow B = E_1E_2...E_kA$
- Post-Multiplication of Elementary matrix $\Leftrightarrow$ Elementary Column Operation

  $A \to B \Leftrightarrow B = AE_1E_2...E_k$
- If E is obtained from I_n by single elementary column operation, then

  $I \xrightarrow{kC_i} E \Leftrightarrow I \xrightarrow{kR_i} E$

  $I \xrightarrow{C_i \leftrightarrow C_j} E \Leftrightarrow I \xrightarrow{R_i \leftrightarrow R_j} E$

  $I \xrightarrow{C_i + kC_j} E \Leftrightarrow I \xrightarrow{R_j + kR_i} E$

2.3 Determinants

Definition (Determinants of 2×2 Matrix).

- Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- $\det(A) = |A| = ad - bc$

Solving Linear equations with determinants for 2×2

- $x_1 = \dfrac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}},\ x_2 = \dfrac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$

Definition (Determinants)**.**

- Suppose A is invertible, then there exists EROs such that
- $A \xrightarrow{ero_1} A_1 \to ... \to A_{k-1} \xrightarrow{ero_k} A_k = I$
- Then det(A) can be evaluated backwards.

  E.g. $A \xrightarrow{R_1 \leftrightarrow R_3} \bullet \xrightarrow{3R_2} \bullet \xrightarrow{R_2+2R_4} I \Rightarrow det(A) = 1 \to 1 \to \frac{1}{3} \to -\frac{1}{3}$
- Let M_{ij} be submatrix where the ith row and jth column are deleted
- Let $A_{ij} = (-1)^{i+j} \det(M_{ij})$, which is the (i, j)-cofactor
- $\det(A) = a_{11}A_{11} + a_{12}A_{12} + ... + a_{1n}A_{1n}$
- $\det(I) = 1$
- $A \xrightarrow{cR_i} B \Rightarrow \det(B) = c \det(A)$

  $I \xrightarrow{cR_i} E \Rightarrow \det(E) = c$
- $A \xrightarrow{R_1 \leftrightarrow R_2} B \Rightarrow \det(B) = -\det(A)$

  $I \xrightarrow{R_1 \leftrightarrow R_2} E \Rightarrow \det(E) = -1$
- $A \xrightarrow{R_i + cR_j} B \Rightarrow \det(B) = \det(A), i \neq j$

  $I \xrightarrow{R_i + cR_j} E \Rightarrow \det(E) = 1$
- $\det(EA) = \det(E) \det(A)$

Calculating determinants easier

- Let A be square matrix. Apply Gaussian Elimination to get REF R
- $A \xrightarrow{E_1} \bullet \xrightarrow{E_2} \bullet ... \bullet \xrightarrow{E_k} R$
- $A \xleftarrow{E_1^{-1}} \bullet \xleftarrow{E_2^{-1}} \bullet ... \bullet \xleftarrow{E_k^{-1}} R$
- Since E_i and E_k^{-1} is type II or III, $\det(E_i) = -1/1$

  $\det(A) = (-1)^t \det(R)$, where t is no of type II or III operations
- If A is singluar, then R has a zero row, and then $det(A) = 0$
- If A is invertible, then all rows of R are nonzero

  $\det(R) = a_{11}a_{22}...a_nn$, the product of diagonal entries.

2.4 Recap

- If A has a REF

  If there is a zero row => Singular matrix

  All rows are nonzero => invertible Matrix

- If A is invertible, Using Gauss Jordan Elim $(A|I) \rightarrow (I|A^{-1})$
-

2.5 More about Determinants

Definition (Determinant Properties).
A is a Square Matrix

- $\det(A) = 0 \Rightarrow A$ is singular
- $\det(A) \neq 0 \Rightarrow A$ is invertible
- $\det(A) = \det(A^T)$
- $\det(cA) = c^n \det(A)$, where n is the order of the matrix
- If A is triangular, $\det(A)$ product of diagonal entries
- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = [\det(A)]^{-1}$

Cofactor Expansion:

- To eavluate determinant using cofactor expansion, expand row/column with most no of zeros.

2.6 Finding Determinants TLDR

Definition (Finding Determinants).

- If A has zero row / column, $\det(A) = 0$
- If A is triangular, $det(A) = a_{11}a_{22}...a_{nn}$
- If Order $n = 2 \rightarrow \det(A) = a_{11}a_{22} - a_{12}a_{21}$
- If row/column has many 0, use cofactor expansion
- Use Gaussian Elimination to get REF
  $$\det(A) = (-1)^t \det(R), t \text{ is no of type } II \text{ operations}$$

Definition (Finding Inverse with Adjoint Matrix).

- $\text{adj}(A) = (A_{ji})_{n\times n} = (A_{ij})^T_{n\times n}$
- $A^{-1} = [\det(A)]^{-1}\text{adj}(A)$

Definition (Cramer's Rule)**.** Suppose A is an invertible matrix of order n

- Liner system $Ax = b$ has unique solution
- $x = \dfrac{1}{\det(A)} \begin{pmatrix} \det(A_1) \\ \vdots \\ det(A_n) \end{pmatrix}$,
- A_j is obtained by replacing the jth column in A with b.

3 Vector Spaces

3.1 Euclidian n-Spaces

Definition (Vector Definitions)**.**

- n-vector : $v = (v_1, v_2, ..., v_n)$
- $\overrightarrow{PQ} // \overrightarrow{P'Q'} \Rightarrow \overrightarrow{PQ} = \overrightarrow{P'Q'}$
- $||\overrightarrow{PQ}|| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$
- $u + v = (u_1 + v_1, u_2 + v_2), u = (u_1, u_2), v = (v_1, v_2)$
- n-vector can be viewed as a row matrix / column matrix

4 Reference

Theorem 4.1. This is a theorem.

Proposition 4.2. This is a proposition.

Principle 4.3. This is a principle.

Note. This is a note

Definition (Some Term)**.** This is a definition