Function and Limits

- $\lim\limits_{x\to\pm\infty} \frac{Ax^\alpha}{Bx^\beta} \begin{cases} 0 & \text{if} \alpha < \beta \\ \frac{A}{B} & \text{if} \alpha = \beta \\ \pm\infty & \text{if} \alpha > \beta \end{cases}$
- $\lim\limits_{x\to c} \frac{sin(g(x))}{g(x)} = 1 (\lim\limits_{x\to c} g(x) = 0)$
- $\lim\limits_{x\to c} \frac{tan(g(x))}{g(x)} = 1$
- $\lim\limits_{x\to 0} \frac{sin(x)}{x} = 1$
- $\lim\limits_{x\to 0} \frac{tan(x)}{x} = 1$
- $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0$

Differentiation

parametric differentiaton: $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$

$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$a^{f(x)}$	$\ln a \cdot f'(x) a^{f(x)}$
$\log_a f(x)$	$\log_a e \cdot \frac{f'(x)}{f(x)}$
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \quad \lvert f(x)\rvert<1$
$\cos^{-1} f(x)$	$-\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \quad \lvert f(x)\rvert<1$
$\tan^{-1} f(x)$	$\frac{f'(x)}{1+[f(x)]^2}$
$\cot^{-1} f(x)$	$-\frac{f'(x)}{1+[f(x)]^2}$
$\sec^{-1} f(x)$	$\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$
$\csc^{-1} f(x)$	$-\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$

Second Derivative Test: $f'(c) = 0, f''(c) < 0$ then local max, $f''(c) > 0$ local min.

L'Hopital's Rule: Given $\lim\limits_{x\to c} f(x)$ and $g(x) = 0/\pm\infty$

$\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \lim\limits_{x\to c} \frac{f'(x)}{g'(x)}$

- Use for $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Trigo Identities

1. $\sec^2 x - 1 = \tan^2 x$
2. $\csc^2 x - 1 = \cot^2 x$
3. $\sin A \cos A = \frac{1}{2}\sin 2A$
4. $\cos^2 A = \frac{1}{2}(1+\cos 2A)$
5. $\sin^2 A = \frac{1}{2}(1-\cos 2A)$
6. $\sin A \cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))$
7. $\cos A \sin B = \frac{1}{2}(\sin(A+B)-\sin(A-B))$
8. $\cos A \cos B = \frac{1}{2}(\cos(A+B)+\cos(A-B))$
9. $\sin A \sin B = \frac{1}{2}(\cos(A+B)-\cos(A-B))$

Integration

$f(x)$	$\int f(x)$
$\tan ax$	$\frac{1}{a}\ln\lvert\sec(ax)\rvert$
$\cot ax$	$\frac{1}{a}\ln\lvert\cot(ax)\rvert$
$\sec ax$	$\frac{1}{a}\ln\lvert\sec(ax)+tan(ax)\rvert$
$\csc ax$	$\frac{1}{a}\ln\lvert\csc(ax)+cot(ax)\rvert$
$\frac{1}{a^2+(x+b)^2}$	$\frac{1}{a}\tan^{-1}(\frac{x+b}{a})$
$\frac{1}{\sqrt{a^2-(x+b)^2}}$	$\sin^{-1}(\frac{x+b}{a})$
$\frac{1}{a^2-(x+b)^2}$	$\frac{1}{2a}\ln\lvert\frac{x+b+a}{x+b-a}\rvert$
$\frac{1}{(x+b)^2-a^2}$	$\frac{1}{2a}\ln\lvert\frac{x+b-a}{x+b+a}\rvert$

Substitution $\int f(g(x))\cdot g'(x)dx = \int f(u)du, u = g(x)$
By Parts $\int uv'dx = uv - \int u'vdx$, order: LIATE: Differentiate to integrate

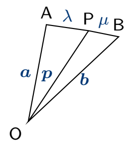
Application of Integration

about x axis

- Vol Disk: $V = \pi\int_a^b f(x)^2 - g(x)^2 dx$
- Vol Shell: $V = 2\pi\int_a^b x\lvert f(x)-g(x)\rvert dx$ (absolute!!)
- Length of curve: $\int_a^b \sqrt{1+f'(x)^2}dx$

Vectors

unit vector: $\hat{p} = \frac{p}{|p|}$, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$



ratio theorem
$\boldsymbol{p} = \frac{\mu\boldsymbol{a}+\lambda\boldsymbol{b}}{\lambda+\mu}$

midpoint theorem
$\boldsymbol{p} = \frac{\boldsymbol{a}+\boldsymbol{b}}{2}$

Dot Product

- $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = |a||b|\cos\theta$
- $a \perp b \Rightarrow a\cdot b = 0$
- $a \parallel b \Rightarrow a\cdot b = |a||b|$

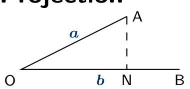
Cross Product

$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_i & a_2 & a_3 \\ b_i & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} (a_2b_3 - a_3b_2) \\ -(a_1b_3 - a_3b_1) \\ (a_1b_2 - a_2b_1) \end{pmatrix}$

$|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$
$a\perp b \Rightarrow a\times b = |a||b|$

$a \parallel b \Rightarrow a\times b = 0$
Parallelogram $= |\boldsymbol{a}\times\boldsymbol{b}|$

Projection



$\triangle ANO = \frac{1}{2}|\overrightarrow{OA}\times\overrightarrow{ON}|$

comp$_{\boldsymbol{b}}\boldsymbol{a} = |\boldsymbol{b}|\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}$
proj$_{\boldsymbol{b}}\boldsymbol{a} =$ comp$_{\boldsymbol{b}}\boldsymbol{a}\cdot\frac{a}{|a|} =$
$\overrightarrow{ON} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}\boldsymbol{b}$

Lines

$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v} = \langle x,y,z\rangle$
$\langle x_0,y_0,z_0\rangle + t\langle a,b,c\rangle$

$\begin{pmatrix} x_0+at \\ y_0+bt \\ z_0+ct \end{pmatrix}$

Planes

$\boldsymbol{n} = \langle a,b,c\rangle, \boldsymbol{r} = \langle x,y,z\rangle, \boldsymbol{r}_0\langle x_0,y_0,c_0\rangle$
Scalar: $\boldsymbol{n}\cdot\boldsymbol{r} = \boldsymbol{n}\cdot\boldsymbol{r}_0$
Cartesian: $ax+by+cz = d$

Distance from Point to Plane

$\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$

Partial Derivatives

Chain Rule

For $z(t) = f(x(t),y(t))$,
$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

For $z(s,t) = f(x(s,t),y(s,t))$,
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$
$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

Arc Length of $r(t)$: $\int_a^b |\boldsymbol{r}'(t)|dt$

Implicit Differentiation

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

Directional Derivative

Gradient vector at $f(x,y): \nabla f = f_x\boldsymbol{i} + f_y\boldsymbol{j}$

$D_u f(x,y) = \langle f_x,f_y\rangle\cdot\langle a,b\rangle = \langle f_x,f_y\rangle\cdot\hat{\boldsymbol{u}} = \nabla f\cdot\hat{\boldsymbol{u}}$ (Unit Vector)

Tangent Plane: $\langle f_x, f_y - 1\rangle\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0$

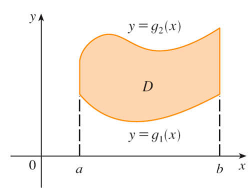
Critical Points

$D = f_{xx}(a,b)f_{yy}(a,b) - (f_{x,y}(a,b))^2$

D	$f_{xx}(a,b)$	**local**
+	+	min
+	-	max
-	any	saddle point
0	any	no conclusion

Double Integrals

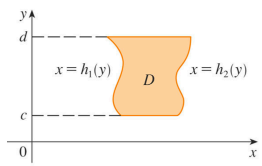
Type I



$\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)dydx$

$D = \{(x,y): a \le x \le b,$
$g_1(x) \le y \le g_2(x)\}$

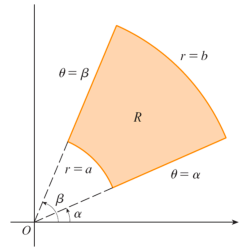
Type II



$\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$

$D = \{(x,y): c \le y \le d,$
$h_1(y) \le x \le h_2(y)\}$

Polar Coordinates



$x = r\cos\theta$
$y = r\sin\theta$
$R = \{(r,\theta): 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$

$\int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$

Surface Area

$S = \iint_R \sqrt{f_x^2 + f_y^2 + 1}dA$, get in the form of $z = f(x,y)$ first

ODE

form	change of variable
$\frac{dy}{dx} = f(x)g(y)$	$\int\frac{1}{g(y)}dy = \int f(x)dx + C$
$y' = g(\frac{y}{x})$	Set $v = \frac{y}{x}$ $\Rightarrow y' = v + xv'$
$y' = f(ax+by+c)$ $\Rightarrow y' = \frac{ax+by+c}{\alpha x+\beta y+\gamma}$	Set $v = ax+by$
$y' + P(x)y = Q(x)$	$R = e^{\int P(x)dx}$ $\Rightarrow y\cdot R = \int Q\cdot Rdx$
$y' + P(x)y = Q(x)y^n$	$z = y^{1-n}$ $\Rightarrow$ sub in Z solve linear

Population Models

$N_\infty = \frac{B}{s}$, $\hat{N} =$ Population Now

Malthus
$N(t) = \hat{N}e^{kt}$
$k = B - D$

Logistic
$\frac{1}{N} = \frac{1}{N_\infty} + (\frac{1}{\hat{N}} - \frac{1}{N_\infty})e^{-Bt}$
$N = \frac{N_\infty}{1+(\frac{N_\infty}{\hat{N}}-1)e^{-Bt}}$

Uranium Decay into Thorium

$U(t) = U_0e^{-k_ut}$, $k = \frac{\ln 2}{\text{halflife}}$, $\frac{dU}{dt} = -k_uU$
Thorium:
$T(t) = \frac{K_uU_0}{K_t-K_u}(e^{-k_ut} - e^{-k_tt})$, $\frac{dT}{dt} = k_uU - k_TT$

Series

Geometric Series

$\sum_{n=1}^{\infty} ar^{n-1}, a \neq 0$ converges to $\frac{a}{1-r}$ when $|r| < 1$, diverges otherwise

If series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Tests

Decreasing function -> differentiate and see the range where $x < 0$

Test	Method
n^{th} term	$\lim_{n\to\infty} a_n \neq 0$ or does not exist, then divergent
Integral	$f(n) = a_n$ is continuous, positive, decreasing function $\forall x \geq 1$ and $\int_1^{\infty} f(x)dx$ converges else divergent
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$convergent $\leftrightarrow p > 1$
Harmonic Series	$\sum_{n=1}^{\infty} \frac{1}{n}$ divergent
Ratio If Factorial	$0 \geq \lim_{n\to\infty} \lvert\frac{a_{n+1}}{a_n}\rvert = L < 1$ abs. convergent, > 1 divergent, $= 1$ inconclusive
Root If nth power	$0 \geq \lim_{n\to\infty} \sqrt[n]{a_n} = L < 1$ abs. convergent, > 1 divergent, $= 1$ inconclusive
Alternating series	b_n decreasing, $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 ...$ is convergent
Power Series	b_n decreasing, $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 ...$ is convergent
Comparison Test	Compare to well known series such as p-series, harmonic

Power Series

$\sum_{n=0}^{\infty} c_n(x-a)^n$ converges at **ONE OF**

- $x = a$
- For all x
- converges if $|x-a| < R$ and diverges if $|x-a| > R$ (R is radius of convergence)

If $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n} = L\right|$ or $\lim_{n\to\infty} \sqrt[n]{|c_n|} = L$, $L \in \mathbb{R}$ or ∞, then $R = \frac{1}{L}$

Taylor and Maclaurin Series

If f has power series repr @ $f = a$, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, |x-a| < R, R > 0$, then $c_n = \frac{f^{(n)}(a)}{n!}$.

Maclaurin Series: $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$ For $-\infty < x < \infty$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

For $-1 < x < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$
$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$$
$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1}$$
$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$
$$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2}$$
$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \ldots$$

Useful Math

- Line: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
- $\int \sqrt{a^2 - x^2}, x = a' \sin\theta, dx = a\cos\theta, = \frac{a^2}{2}\sin^{-1}(\frac{x}{a}) + \frac{x}{2}\sqrt{a^2 - x^2}$
- $\sqrt{a^2 + x^2}dx, x = a\tan\theta, \frac{-\pi}{2} < \theta < \frac{\pi}{2}, = \frac{1}{2}\left(x\sqrt{a^2+x^2} + a^2 \ln\left|\frac{x+\sqrt{a^2+x^2}}{a}\right|\right)$
- $\int \cos^2 x = \frac{1}{4}\sin 2x + \frac{x}{2} = \frac{1}{2}\cos x \sin x + \frac{1}{2}x$
- $\int \sin^2 x = -\frac{1}{4}\sin 2x + \frac{x}{2}$