Vector Spaces and Associated Matrices

Row Space and Column Space

- Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$
- $r_i = (a_{i1} ... a_{in})$
- Row Space of A is the vector space spanned by rows of A : $\operatorname{span}\{r_1, ..., r_m\}$
- $c_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$
- Column space of A is vector space spanned by columns of A : $\operatorname{span}\{c_1, ..., c_n\}$

  Example: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.
  - $r_1 = (1, 2, 3), r_2 = (4, 5, 6)$.
  - $c_1 = (1, 4)^T, c_2 = (2, 5)^T, c_3 = (3, 6)^T$

  Example: To find the basis of A
  - Row Space of A
    - Gaussian elimination on the row-vectors of A and the non-zero rows of R are basis
  - Col space of A
    - Gaussian elimination on row vectors of A and the pivot columns of A are the basis

Row Equivalence

Let A and B be matrices of the same size. A and B are row equivalent if one can be obtained from another by ERO

Theorem: If A and B are row equivalent, then A and B have same row spaces

Remark: Let R be REF of A
- Row space of A = row space of R
- Nonzero rows of R are linearly independent
  - Nonzero rows of R form basis for row space of A
  - No of nonzero rows R = dim of row space of A

Row Operations to Columns

- Let A and B be Row Equivalent Matrices : $A \to A_1 \to ... \to A_{k-1} \to B$
- Exists $E_1, ..., E_k$ s.t. $E_k ... E_1 A = B$
- $M = E_k ... E_1$, M is invertible, $MA = B, A = M^{-1} B$

Theorem: Row Equivalence Preserves the linear relations on the columns
- Suppose $A = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix}$
- $a_j = c_1 a_1 + ... + c_n a_n \Leftrightarrow b_j = c_1 b_1 + ... + c_n b_n$

Remark: Let R be REF of A

- Pivot columns of R form basis for column space of R
- Columns of A which correspond to pivot columns of R form a basis for column space of A
- No of pivot columns of R is the dim of column space of A

Example: Find basis for a vector space V

Let $V = \text{span}\{v_1, v_2, v_3, v_4, v_5, v_6\}, -v_1 = (1, 2, 2, 1), v_2 = (3, 6, 6, 3), \text{etc}$

1. Method 1 (Row Form)
   - View each v_i as a row vector
   - $$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 3 & 6 & 6 & 3 \\ 4 & 9 & 9 & 5 \\ -2 & -1 & -1 & 1 \\ 5 & 8 & 9 & 4 \\ 4 & 2 & 7 & 3 \end{pmatrix} \xrightarrow{\text{Gaussian Elimination}} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
   - V has basis$\{(1, 2, 2, 1), (0, 1, 1, 1), (0, 0, 1, 1)\}$
   - $\dim(V) = 3$
2. Method 2 (Column Form)
   - View each v_i as a column vector
   - $$\begin{pmatrix} 1 & 3 & 4 & -2 & 5 & 4 \\ 2 & 6 & 9 & -1 & 8 & 2 \\ 2 & 6 & 9 & -1 & 9 & 7 \\ 1 & 3 & 5 & 1 & 4 & 3 \end{pmatrix} \xrightarrow{\text{Gaussian Elimination}} \begin{pmatrix} 1 & 3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
   - Using Pivot columns (1,3,5), pick the corresponding vectors, (v_1, v_3, v_5)
   - V has basis$\{(1, 2, 2, 1), (4, 9, 9, 5), (4, 2, 7, 3)\}$
   - $\dim(V) = 3$

Remark: Finding Basis TLDR

1. Method 1: View each $v_1, ..., v_k$ as a row vector
   - Find row echelon form R of $\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$
   - Nonzero rows of R form basis of V
2. Method 2: View each $v_1, ..., v_k$ as a column vector
   - Find row echelon form R of $\begin{pmatrix} v_1 & \cdots & v_k \end{pmatrix}$
   - Find pivot columns of R
   - Corresponding columns v_j form basis for V

Example: Extend S to a basis for $\mathbb{R}^n$

- Find REF of S as row vectors
- Insert the e_i vectors for each i non-pivot column

Theorem: Consistency

Let A be a $m \times n$ matrix

- Column space of A is $\{Av \mid v \in \mathbb{R}^n\}$
- Linear system $Ax = b$ is consistent
  - $\Longleftrightarrow b$ lies in the column space of A

Theorem: Rank

- Let A be a matrix, then $\dim(\text{row space of } A) = \dim(\text{col space of } A)$
- This is the rank of A, denoted by rank(A)

Remark: Let A be an $m \times n$ matrix

- $\text{rank}(A) = \text{rank}(A^T)$
- $\text{rank}(A) = 0 \Leftrightarrow A = 0$
  - $\text{rank}(A) \leq \min\{m, n\}$
  - Full rank if $\text{rank}(A) = \min\{m, n\}$
- Square Matrix A is full rank $\Longleftrightarrow$ A is invertible.

Remark: Rank and Consistency of Linear System

$Ax = b$ be linear system

- $Ax = b$ is consistent
  - $\Leftrightarrow b \in \text{span}\{c_1, ..., c_n\}$
  - $\Leftrightarrow \text{span}\{c_1, ..., c_n\} = \text{span}\{c_1, ..., c_n, b$
  - $\Leftrightarrow \dim(\text{span}\{c_1, ..., c_n\}) = \dim(\text{span}\{c_1, ..., c_n, b\})$
  - $\Leftrightarrow \text{rank}(A) = \text{rank}(A|b)$
- $\text{rank}(A) \leq \text{rank}(A \mid b) \leq \text{rank}(A) + 1$

If $ABx = b$ has a solution, then $Ax = b$ also has a solution

Theorem: Let $A = m \times n$ matrix and $B = n \times p$

- Column space of $AB \subseteq$ column space of A
- Row space of $AB \subseteq$ row space of B
- $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$

Definition: Nullspace and Nullity

- Nullspace of A is the solution space of $Ax = 0$
  - $v \in \mathbb{R}^n \mid Av = 0$
- Vectors in nullspace are viewed as column vectors
- Let R be a REF of A
  - $Ax = 0 \Leftrightarrow Rx = 0$
  - $\text{nullity}(A) = \text{nullity}(R)$

Theorem: Dimension Theorem Let A be a $m \times n$ matrix

- $\text{rank}(A) + \text{nullity}(B) = n$ (n is the number of columns)