MA1522

Linear Algebra in Computing

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Contents

1 Lecture 1

1.1 Linear Algebra

- **Linear** The study of items/planes and objects which are flat
- **Algebra** Objects are not as simple as numbers

1.2 Linear Systems & Their Solutions

Points on a straight line are all the points (x, y) on the xy plane satisfying the linear eqn: $ax + by = c$, where $a, b > 0$

1.2.1 Linear Equation

Linear eqn in n variables (unknowns) is an eqn in the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

where $a_1, a_2, ..., a_n, b$ are constants.

Note. In a linear system, we don't assume that $a_1, a_2, ..., a_n$ are not all 0

- If $a_1 = ... = a_n = 0$ but $b \neq 0$, it is **inconsistent**

  E.g. $0x_1 + 0x_2 = 1$
- If $a_1 = ... = a_n = b = 0$, it is a **zero equation**

  E.g. $0x_1 + 0x_2 = 0$
- Linear equation which is not a zero equation is a **nonzero equation**

  E.g. $2x_1 - 3x_2 = 4$
- The following are not linear equations
  - $xy = 2$
  - $\sin\theta + \cos\phi = 0.2$
  - $x_1^2 + x_2^2 + ... + x_n^2 = 1$
  - $x = e^y$

In the xyz space, linear equation $ax + by + cz = d$ where $a, b, c > 0$ represents a plane

1.2.2 Solutions to a Linear Equation

Let $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ be a linear eqn in n variables
For real numbers $s_1 + s_2 + ... + s_n$, if $a_1s_1 + a_2s_2 + ... + a_ns_n = b$, then $x_1 = s_1, x_2 = s_2, x_n = s_n$ is a solution to the linear equation

The set of all solutions is the **solution set**
Expression that gives the entire solution set is the **general solution**

Zero Equation is satified by any values of $x_1, x_2, ...x_n$
General solution is given by $(x_1, x_2, ..., x_n) = (t_1, t_2, ..., t_n)$

1.2.3 Examples: Linear equation $4x - 2y = 1$

- x can take any arbitary value, say t
- $x = t \Rightarrow y = 2t - \frac{1}{2}$
- General Solution: $\begin{cases} x = t & \text{t is a parameter} \\ y = 2t - \frac{1}{2} \end{cases}$
- y can take any arbitary value, say s
- $y = s \Rightarrow x = \frac{1}{2}s + \frac{1}{4}$
- General Solution: $\begin{cases} y = s & \text{s is a parameter} \\ x = \frac{1}{2}s + \frac{1}{4} \end{cases}$

1.2.4 Example: Linear equation $x_1 - 4x_2 + 7x_3 = 5$

- x_2 and x_3 can be chosen arbitarily, s and t
- $x_1 = 5 + 4s - 7t$
- General Solution: $\begin{cases} x_1 = 5 + 4s - 7t \\ x_2 = s & s, t \text{ are arbitrary parameters} \\ x_3 = t \end{cases}$

1.3 Linear System

Linear System of m linear equations in n variables is

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases} \tag{1}$$

where a_{ij}, b are real constants and a_{ij} is the coeff of x_j in the ith equation

Note. Linear Systems

- If a_{ij} and b_i are zero, linear system is called a **zero system**
- If a_{ij} and b_i is nonzero, linear system is called a **nonzero system**
- If $x_1 = s_1, x_2 = s_2, ..., x_n = s_n$ is a solution to **every equation** in the system, then its a solution to the system
- If every equation has a solution, there might not be a solution to the system
- **Consistent** if it has at least 1 solution
- **Inconsistent** if it has no solutions

1.3.1 Example

$$\begin{cases} a_1x + b1_y = c_1 \\ a_2x + b2_y = c_2 \end{cases} \tag{2}$$

where a_1, b_1, a_2, b_2 not all zero
In xy plane, each equation represents a straight line, L_1, L_2

- If L_1, L_2 are parallel, there is no solution
- If L_1, L_2 are not parallel, there is 1 solution
- If L_1, L_2 coinside(same line), there are infinitely many solution

$$\begin{cases} a_1x + b1_y + c_1z = d_1 \\ a_2x + b2_y + c_2z = d_2 \end{cases} \tag{3}$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ not all zero
In xyz space, each equation represents a plane, P_1, P_2

- If P_1, P_2 are parallel, there is no solution
- If P_1, P_2 are not parallel, there is ∞ solutions (on the straight line intersection)
- If P_1, P_2 coinside(same plane), there are infinitely many solutions
- Same Plane $\Leftrightarrow a_1 : a_2 = b_1 : b_2 = c_1 : c_2 = d_1 : d_2$
- Parallel Plane $\Leftrightarrow a_1 : a_2 = b_1 : b_2 = c_1 : c_2$
- Intersect Plane $\Leftrightarrow a_1 : a_2, b_1 : b_2, c_1 : c_2$ are not the same

1.4 Augmented Matrix

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{1n} & b_1 \\ a_{21} & a_{12} & a_{2n} & b_2 \\ a_{m1} & a_{m2} & a_{mn} & b_m \end{array}\right)$$

1.5 Elementary Row Operations

To solve a linear system we perform operations:

- Multiply equation by nonzero constant
- Interchange 2 equations
- add a constant multiple of an equation to another

Likewise, for a augmented matrix, the operations are on the **rows** of the augmented matrix

- Multiply row by nonzero constant
- Interchange 2 rows
- add a constant multiple of a row to another row

To note: all these operations are revertible

2 Lecture 2

2.1 Recap

Given the linear equation $a_1x_1 + a_2x_2 + ... + a_nx_n = b$

1. $a_1 = a_2 = ... = a_n = b = 0$ zero equation

   Solution: $x_1 = t_1, x_2 = t_2, ... = x_n = t_n$

2. $a_1 = a_2 = ... = a_n = 0 \neq b$ inconsistent

   No Solution

3. Not all $a_1...a_n$ are zero.

   Set $n - 1$ of x_i as params, solve for last variable

2.2 Elementary Row Operations Example

$$\begin{cases} x + \ \ y + 3z = 0 \\ 2x - 2y + 2z = 4 \\ 3x + 9y \qquad = 3 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

2.3 Row Equivalent Matrices

2 Augmented Matrices are row equivalent if one can be obtained from the other by a series of elementary row operations

Given a augmented matrix A, how to find a row equivalent augmented matrix B of which is of a **simple** form?

2.4 Row Echelon Form

Definition (Row Echelon Form (Simple)). Augmented Matrix is in row-echelon form if

- Zero rows are grouped together at the bottom
- For any 2 successive nonzero rows, The first nonzero number in the lower row appears to the right of the first nonzero number on the higher row $\left(\begin{array}{cccc|c} 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$
- Leading entry if a nonzero row is a **pivot point**
- Column of augmented matrix is called
  - **Pivot Column** if it contains a pivot point
  - **Non Pivot Column** if it contains no pivot point
- Pivot Column contains exactly 1 pivot point

  # of pivots = # of leading entries = # of nonzero rows

Examples of row echlon form:

$$\left(\begin{array}{cc|c} 3 & 2 & 1 \end{array}\right) \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right) \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{array}\right) \left(\begin{array}{cccc|c} 0 & 1 & 2 & 8 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Examples of NON row echlon form:

$$\left(\begin{array}{cc|c} 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 \end{array}\right) \left(\begin{array}{cc|c} 0 & 0 & \mathbf{1} \\ \mathbf{1} & -1 & 0 \\ 0 & 0 & 1 \end{array}\right) \left(\begin{array}{ccc|c} \mathbf{1} & 0 & 2 & 1 \\ 0 & \mathbf{1} & 0 & 2 \\ 0 & \mathbf{1} & 1 & 3 \end{array}\right) \left(\begin{array}{cccc|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

2.5 Reduced Row-Echelon Form

Definition (Reduced Row-Echelon Form). Suppose an augmented matrix is in row-echelon form. It is in **reduced row-echelon form** if

- Leading entry of every nonzero row is 1

  Every pivot point is one
- In each pivot column, except the pivot point, all other entries are 0.

Examples of reduced row-echelon form:

$$\begin{pmatrix}1 & 2 & \bigm| & 3\end{pmatrix} \begin{pmatrix}0 & 0 & \bigm| & 0\\0 & 0 & \bigm| & 0\end{pmatrix} \begin{pmatrix}1 & 0 & \bigm| & 0\\0 & 1 & \bigm| & 0\\0 & 0 & \bigm| & 1\end{pmatrix} \begin{pmatrix}1 & 0 & 0 & \bigm| & 1\\0 & 1 & 0 & \bigm| & 2\\0 & 0 & 1 & \bigm| & 3\end{pmatrix} \begin{pmatrix}0 & 1 & 2 & 0 & \bigm| & 1\\0 & 0 & 0 & 1 & \bigm| & 3\\0 & 0 & 0 & 0 & \bigm| & 0\\0 & 0 & 0 & 0 & \bigm| & 0\end{pmatrix}$$

Examples of row-echelon form but not reduced: (pivot point is not 1 / all other elements **in pivot column** must be zero)

$$\begin{pmatrix}\mathbf{3} & 2 & \bigm| & 1\end{pmatrix} \begin{pmatrix}1 & \mathbf{-1} & \bigm| & 0\\0 & 1 & \bigm| & 0\end{pmatrix} \begin{pmatrix}\mathbf{2} & 0 & \bigm| & 0\\0 & 1 & \bigm| & 0\\0 & 0 & \bigm| & 1\end{pmatrix} \begin{pmatrix}\mathbf{-1} & 2 & 3 & \bigm| & 4\\0 & 1 & 1 & \bigm| & 2\\0 & 0 & 2 & \bigm| & 3\end{pmatrix} \begin{pmatrix}0 & 1 & 2 & \mathbf{8} & \bigm| & 1\\0 & 0 & 0 & \mathbf{4} & \bigm| & 3\\0 & 0 & 0 & 0 & \bigm| & 0\\0 & 0 & 0 & 0 & \bigm| & 0\end{pmatrix}$$

To note: 2nd matrix has -1 in the pivot column, but 5th matrix has 2 in a non-pivot column so its fine

2.6 Solving Linear System

If Augmented Matrix is in reduced row-echelon form, then solving it is easy

$$\begin{pmatrix}1 & 0 & 0 & \bigm| & 1\\0 & 1 & 0 & \bigm| & 2\\0 & 0 & 1 & \bigm| & 3\end{pmatrix} \text{ then } x_1 = 1, x_2 = 2, x_3 = 3$$

Note.
- If any equations in the system is inconsistent, the whole system is inconsistent

2.6.1 Examples

Augmented Matrix: $\begin{pmatrix}1 & -1 & 0 & 3 & \bigm| & -2\\0 & 0 & 1 & 2 & \bigm| & 5\\0 & 0 & 0 & 0 & \bigm| & 0\end{pmatrix}$

- The zero row can be ignored. $\begin{cases}x_1 - x_2 \quad + 3x_4 = -2\\ \qquad x_3 + 2x_4 = 5\end{cases}$
- Degree of freedom(# cols): 4, number of restrictions (# pivot cols): 2, arbitrary vars(# non pivot cols): 4-2 = 2. Set this to the non-pivot cols

1. Let $x_4 = t$ and sub into 2nd eqn

   $x_3 + 2t = 5 \Rightarrow x_3 = 5 - 2t$

2. sub $x_4 = t$ into 1st eqn

   $x_1 - x_2 + 3t = -2$

   Let $x_2 = s$. Then $x_1 = -2 + s - 3t$

3. Infinitely many sols with (s and t as arbitrary params)

   $x_1 = -2 + s - 3t, x_2 = s, x_3 = 5 - 2t, x_4 = t$

Augmented Matrix: $\left(\begin{array}{ccccc|c} 0 & 2 & 2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{array}\right)$

- $\begin{cases} 0x_1 + 2x_2 + 2x_3 + 1x_4 - 2x_5 = 2 \\ x_3 + \ x_4 + \ x_5 = 3 \\ 2x_5 = 4 \end{cases}$

- Degree of freedom: 5, number of restrictions: 3, arbitrary vars: 5-3 = 2

1. by 3rd eqn, $2x_5 = 4 \Rightarrow x_5 = 2$

2. sub $x_5 = 2$ into 2nd eqn

   $x_3 + x_4 + 2 = 3 \Rightarrow x_3 + x_4 = 1$

   let $x_4 = t$. Then $x_3 = 1 - t$

3. sub $x_5 = 2, x_3 = 1 - t, x_4 = t$ into 1st eqn

   $2x_2 + 2(1 - t) + t - 2(2) = 2 \Rightarrow 2x_2 - t = 4 \Rightarrow x_2 = \frac{t}{2} + 2$

4. system has inf many solns: $x_1 = s, x_2 = \frac{t}{2} + 2, x_3 = 1 - t, x_4 = t, x_5 = 2$ where s and t are arbitrary

2.6.2 Algorithm

Given the augmented matrix is in row-echelon form.

1. Set variables corresponding to non-pivot columns to be arbitrary parameters

2. Solve variables corresponding to pivot columns by back substitution (from last eqn to first)

2.7 Gaussian Eliminiation

Definition (Gaussian Elimination)**.**

1. Find the left most column which is not entirely zero

2. If top entry of such column is 0, replace with nonzero number by swapping rows

3. For each row below top row, add multiple of top row so that leading entry becomes 0

4. Cover top row and repeat to remaining matrix

Note (Algorithm with Example)**.**

$$\left(\begin{array}{cccccc|c} 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right)$$

1. Find the left most column which is not all zero (2nd column)

2. Check top entry of the selection. If its zero, replace it by a nonzero number by interchanging the top row with another row below

$$\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right)$$

3. For each row below the top row, adda suitable multiple of top row so that leading entry becomes 0.

   $2R_1 + R_3$ will ensure that the -2 turns to 0

$$\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 3 & 6 & 9 & -12 \end{array}\right)$$

4. Cover top row and repeat procedure to the remaining matrix

$$\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ \hline 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 3 & 6 & 9 & -12 \end{array}\right)$$

   Look at C_4. $R_3 \times -1.5R_2$ will set R_3C_4 to zero.

$$\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ \hline 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right)$$

   This is now in row echelon form.

Only use $R_i \Leftrightarrow R_j or R_i + CR_j$ in this method.

2.8 Gauss-Jordan Elimination

Definition (Gauss Joran Elimination)**.**

1-4. Use Gaussian Eliminiation to get row-echelon form

5. For each nonzero row, multiply a suitable constant so pivot point becomes 1

6. Begin with last nonzero row and work backwords

   Add suitable multiple of each row to the rows above to introduce 0 above pivot point

- Every matrix has a unique reduced row-echelon form.

- Every nonzero matrix has infinitely many row-echelon ofrm

Note (Gauss Jordan Elimination Example)**.** Suppose an augmented matrix is in row-echelon form. $\left(\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right)$

1. All pivot points must be 1

   multiply R_2 by $\frac{1}{2}$ and R_3 by $\frac{1}{6}$

   $$\left(\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$$

2. In each pivot col, all entries other than pivot point must be 0. Work backwards

   $R_1 + -3R_1$, $R_2 + -R_1$

   $$\left(\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$$

   $R_1 + -4R_2$

   $$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 0 & -29 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$$

3 Lecture 3

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4 Lecture 4

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5 Lecture 5

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6 Lecture 6

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7 Reference

Theorem 7.1. This is a theorem.

Proposition 7.2. This is a proposition.

Principle 7.3. This is a principle.

Note. This is a note

Definition (Some Term)**.** This is a definition