- **Mutually Exclusive** - $A \cap B = \emptyset$
- **Union** - $A \cup B = \{x : x \in A \vee x \in B\}$
- **Intersection** - $A \cap B = \{x : x \in A \wedge x \in B\}$
- **Complement** - $A' = \{x : x \in S \wedge x \notin A\}$
- $(A \cap B)' = (A' \cup B')$
- **Multiplication** - R experiments performed sequentially. Then $n_i \cdot ... \cdot n_r$ possible outcomes for r experiments
- **Addition** - e can be performed k ways, and k ways do not overlap : total ways: $n_1 + ... + n_k$
- **Permutation** - Arrangement of r objects out of n, *ordered*. $P_r^n = \frac{n!}{(n-r)!}, P_n^n = n!$
- **Combination** - Selection of r objects out of n, *unordered* $\binom{n}{r} = \frac{n!}{r!(n-r)!}, \binom{n}{r} \times P_r^r = P_r^n$

Probability

- Axioms:
  1. $0 \le P(A) \le 1$
  2. $P(S) = 1$
- Propositions:
  1. $P(\emptyset) = 0$
  2. $A_1 ... A_n$ are mutually exclusive,$P(A_1 \cup ... \cup A_n) = P(A_1) + ... + P(A_n)$
  3. $P(A') = 1 - P(A)$
  4. $P(A) = P(A \cap B) + P(A \cap B')$
  5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  6. If $A \subset B, P(A) \le P(B)$

Conditional Probability

- $P(B|A)$ is probability of B given that A has occured
- $P(B|A) = \frac{P(A \cap B)}{P(A)}$
- $P(A \cap B) = P(B|A)P(A)$
- $P(A|B) = \frac{P(A)P(B|A)}{P(B)}$
- $P(A \cap B \cap C) = P(A)P(B|A)P(C|B \cap A)$
- **Independent** - $P(A \cap B) = P(A)P(B), A \perp B$
  - If $P(A) \ne 0, A \perp B \leftrightarrow P(B|A) = P(B)$ (Knowledge of A does not change B)
- **Independence vs Mutually exclusive** -
  - $P(A) > 0 \wedge P(B) > 0, A \perp B \Rightarrow$ not mutually exclusive
- Partition - $A_i ... A_n$ is mutually exclusive and $\bigcup_i^n = 1 A_i = S, A_i ... A_n$ is partition of S
  - $P(B) = \sum_{i=1}^n P(B \cap A_i) = \sum_{i=1}^n P(A_i)P(B|A_i)$
  - $n = 2, P(B) = P(A)P(B|A) + P(A')P(B|A')$
- **Bayes Theorem** - $P(A_k|B) = \frac{P(A_k)P(B|A_k)}{\sum_{i=1}^n P(A_i)P(B|A_i)}$
- $n = 2, P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A)+P(A')P(B|A')}$

Random Variables

- Notations:
  - $\{X = x\} = \{s \in S : X(s) = x\} \in S$
  - $\{X \in A\} = \{s \in S : X(s) \in A\} \in S$

Probability Distributions

- PMF(*Discrete*) of $X - f(x) = \begin{cases} P(X=x) \text{ if } x \in R_X \\ 0 \text{ otherwise} \end{cases}$
- Properties (**must** satisfy)
  1. $f(x_i) \ge 0, x_i \in R_X$
  2. $f(x_i) = 0, x_i \notin R_X$
  3. $\sum_{i=1}^{\infty} f(x_i) = 1$
- PDF(*Continuous*) of X is function that satisfies the following
  1. $f(x) \ge 0, x \in R_X$ and $f(x) = 0, x \notin R_X$
  2. $\int_{R_X} f(x)\,\mathrm{d}x = 1$
  3. $a \le b, P(a \le X \le b) = \int_a^b f(x)\,\mathrm{d}x$
  - To validate, check (1) and (2)
- CDF (Discrete) - $F(X) = P(X \le x)$
  - $P(a \le X \le b) = P(X \le b) - P(X < a) = F(b) - F(a-), a-$ (is largest value in R_X smaller than a)
- CDF(Continuous) - $F(X) = \int_{-\infty}^x f(t)\,\mathrm{d}t$
  - $f(x) = \frac{\mathrm{d}}{\mathrm{d}t}F(x)$
  - $P(a \le X \le b) = F(b) - F(a)$
  - $F(x)$ is non-decreasing.
  - PDF/PMF and CDF have 1 to 1 correspondence.
  - Ranges of $F(x)$ and $f(x)$ satisfy:
    - $0 \le F(X) \le 1$
    - for discrete: $0 \le f(X) \le 1$
    - for continuous: $f(x) \ge 0$, not necessary $f(x) \le 1$

Expectation

- Expectation(Discrete):

$$E(X) = \mu_X = \sum_{x_i \in R_X} x_i f(x_i)$$

- Expectation(Continuous):

$$E(X) = \mu_X = \int_{-\infty}^{\infty} x_i f(x_i)$$

- **Properties**
  1. $E(aX + b) = aE(X) + b$
  2. $E(X + Y) = E(X) + E(Y)$
  3. Let $g(\cdot)$ be arbitrary function.
     - $$E[g(X)] = \sum g(x)f(x)$$ or $$E[g(X)] = \int_{R_X} g(x)f(x)$$
     - example - $E(X^2) = \sum x^2 f(x)$
- Variance -

$$\sigma_x^2 = V(X) = E(X - \mu)^2 = E(X^2) - E(X)^2$$

  - Discrete - $V(X) = \sum (x - \mu_x)^2 f(x)$
  - Continuous - $V(X) = \int_{-\infty}^{\infty} (x - \mu_x)^2 f(x)$
  - **Properties**
    1. $V(aX + b) = a^2 V(X)$
    2. $V(X) = E(X^2) - E(X)^2$
    3. Standard Deviation = $\sigma_x = \sqrt{V(X)}$

Joint Probability Function

- Discrete

$$f_{X,Y}(x, y) = P(X = x, Y = y)$$

- **Properties**
  1. $f(X, Y)(x, y) \ge 0, (x, y) \in R_{X,Y}$
  2. $f(X, Y)(x, y) = 0, (x, y) \notin R_{X,Y}$
  3. $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (x_i, y_i) = 1$
- Continuous

$$P((X, Y) \in D) = \iint_{(x,y) \in D} f(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

  - $P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x$
- **Properties**
  1. $f_{X,Y}(x, y) \ge 0$, for any $(x, y) \in R_{X,Y}$
  2. $f_{X,Y}(x, y) = 0$, for any $(x, y) \notin R_{X,Y}$
  3. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}x\,\mathrm{d}y = 1$

Marginal Probability Distribution

- Discrete - $f_X(x) = \sum_y f_{X,Y}(x, y)$
- Continuous - $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}y$
- Conditional Distribution -

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

  - If $f_X(x) > 0, f_{X,Y}(x, y) = f_X(x) f_{Y|X}(y|x)$
  - $P(Y \le y \mid X = x) = \int_{-\infty}^y f_{Y|X}(y|x)\,\mathrm{d}y$
  - $E(Y \mid X = x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\,\mathrm{d}y$